ELSEVIER
Journal of Pure and Applied Algebra 102 (1995) 273–287
JOURNAL OF PURE AND APPLIED ALGEBRA

Rings for which certain modules are CS

Nguyen V. Dung[a,*], Patrick F. Smith[b]

[a] Institute of Mathematics, P. O. Box 631 Bo Ho, Hanoi, Viet Nam
[b] Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland, United Kingdom

Communicated by C. A. Weibel; received 7 October 1992; revised 3 January 1994

Abstract

If $\mathbb{C}$ is any class of modules over a general ring R such that $\mathbb{C}$ is closed under direct sums, quotients and submodules, then every module in $\mathbb{C}$ is CS if and only if every module M in $\mathbb{C}$ has a decomposition $M = \oplus_{i \in I} M_i$, where each module M_i ($i \in I$) is either simple, or has length 2 and is X-injective for each module X in $\mathbb{C}$. In consequence, necessary and sufficient conditions are given for a ring to have all its right singular modules CS. Rings whose finitely generated modules are CS are also studied.

AMS Classification: Primary 16D70; Secondary 16D50, 16D40, 16E50, 16P60

0. Introduction

A module M is called a CS-module if every submodule of M is essential in a direct summand of M. CS-modules occur naturally in several contexts, and various authors have studied rings over which certain classes of modules are CS. For example, Goodearl [14, 15] studied right nonsingular rings all of whose nonsingular right modules are projective, and as is easily seen, these are precisely the right nonsingular rings over which every projective right module is CS. Oshiro [24, 25] studied rings with the latter property without assuming the nonsingularity condition, and he called such rings right co-H-rings. CS-modules and CS-rings have been investigated extensively also by Camillo–Yousif [4], Chatters–Hajarnavis [5], Chatters–Khuri [7], Harada [17], Kamal–Müller [19, 20] and Okado [23].

In [28], Osofsky and Smith proved that a cyclic module M has finite uniform dimension if all quotients of cyclic submodules of M are CS (see also [10]). The purpose of this paper is to continue the investigation of CS-modules and to answer some open questions which were raised in [28].

*Corresponding author.

0022-4049/95/$09.50
SSDI 0022-4049(95)00084-V

Goodearl [14] characterized right SI-rings, i.e. rings whose singular right modules are injective. The question was raised in [28, p.351] of characterizing the rings whose singular right modules are CS. In fact, this question leads us to a rather more general study. We will show that if $\mathbb{C}$ is any class of modules (over any ring) which is closed under direct sums, quotients and submodules, then every module M in $\mathbb{C}$ is CS if and only if every module M in $\mathbb{C}$ has a decomposition $M = \bigoplus_{i \in I} M_i$, where each M_i is either simple, or has length 2 and is X-injective for every $X \in \mathbb{C}$ (Theorem 7). Using this result we obtain necessary and sufficient conditions for a general ring to have all singular right modules CS (Corollary 8) and all right modules CS (Theorem 11).

Next we study the question when all finitely generated right modules over a ring R are CS (cf. [28], remarks after Corollary 1). If every finitely generated right R-module is quasi-continuous, the ring R must be semisimple (see [26] and [28, p. 348]). We will show that for some large classes of rings R, if all finitely generated (or even 2-generated) right R-modules are CS then all right R-modules are CS (Theorem 11 and Corollary 14). This is not true in general if R is right nonsingular (Theorems 11 and 13).

1. Definitions and notation

Throughout this paper, we consider only associative rings with identity, and unless otherwise stated, all modules will be unitary right modules.

Let R be any ring. A submodule N of an R-module M is called a *complement* submodule of M if N has no proper essential extensions in M. Recall that a module M is called a *CS-module* (or simply *CS*) provided every complement submodule of M is a direct summand of M, or equivalently, every submodule of M is a essential in a direct summand of M. A module M is called *quasi-continuous* if M is CS and for any direct summands A and B of M with $A \cap B = 0$, $A \oplus B$ is also a direct summand of M.

A nonempty family $\{A_i : i \in I\}$ of submodules of a module M is called a *local direct summand* of M if $\Sigma_I A_i$ is a direct and $\Sigma_F A_i$ is a direct summand of M for any finite subset $F \subseteq I$. By a *locally semi-T-nilpotent* family of modules is meant a family $\{M_i : i \in I\}$ such that, for any countable set of nonisomorphisms

$$\{f_n : M_{i(n)} \to M_{i(n+1)}\},$$

with $i(n) \neq i(m)$ in I, for $n \neq m$, and for any $x \in M_{i(1)}$, there exists k (depending on x) such that $f_k \ldots f_2 f_1(x) = 0$ (see [17, p. 174]).

For any module M, we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are all submodules of M-generated modules. In other words, $N \in \sigma[M]$ if and only if N is a submodule of a quotient of a direct sum of copies of M. It is well known that $\sigma[M]$ is a locally finitely generated Grothendieck category (see, for example, [33]).

Let $\mathbb{C}$ be a Grothendieck category. A short exact sequence $O \to X \to Y \to Z \to O$ in $\mathbb{C}$ is called a *pure sequence* when the induced morphism $p: \mathrm{Hom}_{\mathbb{C}}(F, Y) \to \mathrm{Hom}_{\mathbb{C}}(F, Z)$ is an epimorphism for every finitely presented object F *of* $\mathbb{C}$. In this case X is called a pure *subobject of* Y. An object E of $\mathbb{C}$ is called *pure-injective* when it has the injectivity property with respect to all pure sequences in $\mathbb{C}$. A locally finitely presented Grothendieck category $\mathbb{C}$ is called *pure semisimple* if each of its objects is pure-injective.

A module M is called *uniserial* provided for any submodules A and B of M, either $A \subseteq B$ or $B \subseteq A$. *A ring R* is called *right serial* if R is a direct sum of uniserial right ideals. *Left serial* rings are defined similarly. A module M is called a *V-module* if every simple right R-module is M-injective.

For any module M, $J(M)$, $\mathrm{Soc}(M)$, $Z(M)$ and $E(M)$ will denote the Jacobson radical, the socle, the singular submodule and the injective hull of M, respectively. For a module M of finite length, the composition length of M is denoted by *length* M.

2. Preliminary lemmas

Throughout this section R is a general ring and M an R-module. To prove our main results, we shall require a number of preparatory lemmas. The first lemma follows easily from [28, Theorem 1].

Lemma 1. *Let M be a finitely generated module such that every quotient of every cyclic submodule of M is CS. Then every quotient of M has finite uniform dimension.*

Proof. By [28, Theorem 1], every quotient of every cyclic submodule of M has finite uniform dimension. An easy induction on the number of generators of M gives that M has finite uniform dimension. Now apply the same argument to the quotients of M. □

Our next lemma extends somewhat a result of Okado [23] that any CS-module over a right Noetherian ring is a direct sum of uniform modules. A module M is called *locally Noetherian* if every finitely generated submodule of M is Noetherian.

Lemma 2. *Let M be a locally Noetherian CS-module. Then every local direct summand of M is a direct summand, and M is a direct sum of uniform modules.*

Proof. Let $m \in M$. Let $\mathrm{r}(m) = \{r \in R\colon mr = 0\}$. Then $R/\mathrm{r}(m) \cong mR$, so $R/\mathrm{r}(m)$ is Noetherian. Now apply [22, Theorem 2.17 and Proposition 2.18]. □

If the direct sum of two modules is quasi-continuous, then these modules are relatively injective (see, for example, [22, Proposition 2.10]). This fails for a CS direct sum, but we have the following which is still very useful when dealing with CS-modules. It could be deduced from [2, Lemma 8], but we give a short direct proof here for completeness.

Lemma 3. *Let A and B be uniform modules with local endomorphism rings such that $M = A \oplus B$ is CS. Let C be a submodule of A and let $\theta: C \to B$ be a homomorphism. Then the following hold.*

(a) *If θ is not extended to a homomorphism from A to B, then θ is a monomorphism and B is embedded in A.*

(b) *If any monomorphism $\varphi: B \to A$ is an isomorphism, then B is A-injective.*

(c) (cf. [20, Lemma 13]) *If B is not embedded in A, then B is A-injective.*

Proof. (a) Suppose θ cannot be extended to A. Let $U = \{(x, -\theta(x)) : x \in C\} \subseteq A \oplus B$. Then U is a submodule of M and clearly $U \cap B = 0$. Since M is CS, there is a direct summand U^* of M such that U is essential in U^*. By the Krull–Schmidt–Azumaya theorem (see, for example, [1, Corollary 12.7]), we have $M = U^* \oplus A$ or $M = U^* \oplus B$.

Suppose that $M = U^* \oplus B$. Let $\pi: U^* \oplus B \to B$ be the projection. Then it is easy to see that $\pi|_A$ extends $\theta: C \to B$, a contradiction. Thus $M = U^* \oplus A$ which implies that $\theta(x) \neq 0$ for $x \neq 0$, i.e. θ is a monomorphism. Since $U^* \cap B = 0$, clearly B is embedded in A.

(b) As in the proof of (a), given any homomorphism $\theta: C \to B$ with $C \subseteq A$, suppose that $M = U^* \oplus A$. Let $\psi: U^* \oplus A \to A$ be the projection. Then clearly $\psi|_B$ is a monomorphism (because U is essential in U^*), hence an isomorphism by hypothesis. It follows easily that $M = U^* \oplus B$, so, as in (a), θ can be extended to a homomorphism from A to B. It follows that B is A-injective.

(c) Immediate by (a). $\square$

Lemma 4. (Osofsky [27, Lemma B]). *Let M be a uniserial module with unique composition series $M \supset U \supset V \supset 0$. Then $M \oplus (U/V)$ is not a CS-module.*

Proof. Clearly M and U/V have local endomorphism rings. Suppose that $M \oplus (U/V)$ is CS. Let $\pi: U \to U/V$ be the canonical homomorphism. Since π is not a monomorphism, by Lemma 3(a), π can be extended to a homomorphism $\varphi: M \to U/V$. Since U/V is simple, Ker $\varphi = U$ or M, a contradiction. $\square$

This lemma shows that the direct sum of a uniserial module of length 3 and a simple module need not be CS. However, the direct sum of a module of length 2 and a simple module is always CS. In fact, the following more general result holds. The proof uses some techniques from Kamal–Müller [20].

Lemma 5. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules $M_i (i \in I)$, each of composition length at most 2. Suppose further, that M_j is M_k-injective for all $j \neq k \in I$ with M_j, M_k both of length 2. Then M is a CS-module.*

Proof. First we show that every maximal uniform submodule of M is a direct summand of M. Let D be any maximal uniform submodule of M. Let $0 \neq x \in D$. Then there exists a finite subset I' of I such that $x \in \bigoplus_{i \in I'} M_i$. Since xR is essential in D, it is easy to see that D can be embedded in $\bigoplus_{i \in I'} M_{i'}$ and hence D is finitely generated. There exist a positive integer n and $i(j) \in I$ $(1 \leq j \leq n)$ such that $D \subseteq M_{i(1)} \oplus \cdots \oplus M_{i(n)} = N$, and we choose n minimal.

For $1 \leq j \leq n$, $\pi_j : N \to M_{i(j)}$ denotes the projection. Since $\bigcap_{1 \leq j \leq n} \operatorname{Ker}(\pi_j I_{\mathrm{D}}) = 0$ and D is uniform, without loss of generality, we can suppose that $\operatorname{Ker}(\pi_1 I_{\mathrm{D}}) = 0$ and hence $D \cap (M_{i(2)} \oplus \cdots \oplus M_{i(n)}) = 0$. Thus D can be embedded in $M_{i(1)}$, so D is simple or has length 2.

Suppose first that D has length 2. Then $\pi_1(D) = M_{i(1)}$ and hence $N = D \oplus M_{i(2)} \oplus \cdots \oplus M_{i(n)}$. Now suppose that D is simple. By the choice of n, $\pi_j(D) \neq 0$ $(1 \leq j \leq n)$. Suppose there exists $1 \leq k \leq n$ such that $M_{i(k)}$ is simple. Then $\pi_k(D) = M_{i(k)}$, and hence $N = D \oplus (\bigoplus_{j \neq k} M_{i(j)})$. Otherwise, length $M_{i(j)} = 2$ $(1 \leq j \leq n)$, and, by hypothesis, N is N-injective, and hence CS. Thus, D is a direct summand of N, and hence also of M.

Now we claim that any complement submodule C of M contains a nonzero uniform direct summand of M. Indeed, there is a nonzero uniform submodule K in C. Then K has a maximal essential extension K' in C. Clearly K' is a complement submodule of C, and since C is a complement submodule of M, K' is a complement submodule of M (see [5, Proposition 2.2]). Because K' is uniform, K' is a direct summand of M, by the above argument.

Now let A be any complement submodule of M. By Zorn's Lemma, there exists a maximal local direct summand $\{A_\alpha : \alpha \in \Omega\}$ of M such that $A_\alpha \subseteq A$ and A_α is uniform for all $\alpha \in \Omega$. Since End M_i is local and length $M_i \leq 2$ for each $i \in I$, the family $\{M_i : i \in I\}$ is locally semi-T-nilpotent by [18, Lemma 12], and hence every local direct summand of M is a direct summand (see [17, Theorem 7.3.15]). Thus $\bigoplus_{\alpha \in \Omega} A_\alpha$ is a direct summand of M. Now $A = (\bigoplus_\Omega A_\alpha) \oplus B$ for some submodule B of A. If $B \neq 0$, again by [5, Proposition 2.2], B is a complement submodule of M, hence B contains a nonzero uniform direct summand A' of M. Then $\{\{A_\alpha : \alpha \in \Omega\}, A'\}$ is a local direct summand of M, which contradicts the maximality of $\{A_\alpha : \alpha \in \Omega\}$. Thus $B = 0$, and $A = \bigoplus_\Omega A_\alpha$ is a direct summand of M. Therefore M is a CS-module, and the proof is complete. $\square$

3. Classes of modules

Again R is a general ring and M is an R-module. The next lemma gives, for a given module M, a sufficient condition for the category $\sigma[M]$ to be a pure semisimple category.

Lemma 6. *Let M be a module and suppose that for each module N in $\sigma[M]$, every local direct summand of N is a direct summand of N. Then $\sigma[M]$ is a pure semisimple category.*

Proof. By a result of Simson [30, Theorem 1.9], a locally finitely presented Grothendieck category $\mathbb{C}$ is pure semisimple if and only if the direct sum of any family of pure-injective objects in $\mathbb{C}$ is pure-injective. Note first that, by hypothesis, $\sigma[M]$ is locally finitely presented. For, every module in $\sigma[M]$ is a direct sum of indecomposable modules, so that $\sigma[M]$ is locally Noetherian, and hence locally finitely presented. Let $\{A_i\colon i \in I\}$ be any family of pure-injective objects in $\sigma[M]$, and let $A = \bigoplus_I A_i$. Let P be the (categorical) direct product of $\{A_i\colon i \in I\}$ in $\sigma[M]$. Since $\sigma[M]$ is a Grothendieck category, P always exists and in fact is the largest submodule of the usual direct product $\prod_I A_i$ (in Mod-R) which belongs to $\sigma[M]$ (see, for example, [33, 15.1, 13.5]). A standard argument shows that P is also a pure-injective object in $\sigma[M]$. Clearly P contains A as a local direct summand. By the hypothesis, A is a direct summand of P, hence A is pure-injective in $\sigma[M]$. Thus $\sigma[M]$ is a pure semisimple category. □

We are now in a position to prove our main result.

Theorem 7. *Let R be any ring and let $\mathbb{C}$ be any class of R-modules which is closed under direct sums, quotients and submodules. Then the following statements are equivalent.*

(a) *Every module M in $\mathbb{C}$ is a CS-module.*

(b) *Every module M in $\mathbb{C}$ has a decomposition $M = \bigoplus_{i \in I} M_i$, where length $M_i \leq 2$ for each $i \in I$, and if length $M_j = 2$, for some $j \in I$, then M_j is X-injective for every module X in $\mathbb{C}$.*

(c) *Every (cyclic) module M in $\mathbb{C}$ is a direct sum $M = N \oplus S$ of a semisimple submodule S and a submodule N such that N is X-injective for all X in $\mathbb{C}$.*

Proof. (a) ⇒ (b). Suppose that every module in $\mathbb{C}$ is *CS*. We will proceed in two steps.

Step 1: Every finitely generated module M in $\mathbb{C}$ is Noetherian.

Let $M \in \mathbb{C}$ with M finitely generated. Suppose first that $\mathrm{Soc}(M) = 0$. By Lemma 1, M is a finite direct sum of uniform modules, so without loss of generality, we may assume that M is uniform. Clearly $\sigma[M] \subseteq \mathbb{C}$. Let $\hat{M}$ be the injective hull of M in $\sigma[M]$; then $\hat{M} \in \mathbb{C}$, $\hat{M}$ is quasi-injective and M is essential in $\hat{M}$ (see, for example, [33, 17.9]).

Let T be any simple module which is a quotient of a submodule of $\hat{M}$. Then $T \in \mathbb{C}$ and $\hat{M} \oplus T$ is *CS* by hypothesis. Now End $\hat{M}$ and End T are local, and since $\mathrm{Soc}(\hat{M}) = 0$, T is not embedded in $\hat{M}$. Thus, by Lemma 3, T is $\hat{M}$-injective. It follows that $\hat{M}$ is a V-module, so M is also a V-module. By Lemma 1, every quotient of M has finite uniform dimension. Thus M is Noetherian by [11, Corollary 3].

Now let M be any finitely generated module in $\mathbb{C}$, and suppose that M is not Noetherian. As before, we can suppose that M is uniform. By the above argument, M has a nonzero simple socle A_1. Again, if $\mathrm{Soc}(M/A_1) = 0$, then M/A_1 is Noetherian, a contradiction. Let A_2 be the submodule of M such that $A_2/A_1 = \mathrm{Soc}(M/A_1)$. Then $A_2 \neq A_1$ and, by Lemma 1, M/A_1 has finite uniform dimension, so A_2 is of finite length. By induction, we obtain a strictly ascending sequence

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subseteq M,$$

with $A_{n+1}/A_n = \mathrm{Soc}(M/A_n)$ $(n \geq 0)$. Set $A = \bigcup_{n\geq 1} A_n$; then because each A_n is of finite length, A is locally Noetherian. Then every module $L \in \sigma[A]$ is locally Noetherian (see [33, 27.3]) and CS, so that, by Lemma 2, every local direct summand of L is also a direct summand. By Lemma 6, $\sigma[A]$ is a pure semisimple category, so every module in $\sigma[A]$ is a direct sum of Noetherian modules (see [30] or [33, 53.4, 53.5]). Thus, because A is uniform, A must be Noetherian. But in this case the ascending chain $0 \subset A_1 \subset A_2 \subset \cdots \subseteq A$ cannot be infinite, a contradiction. This shows that M is Noetherian.

Step 2: Every module M in $\mathbb{C}$ is a direct sum of modules of length at most 2.

Let $M \in \mathbb{C}$ and consider the category $\sigma[M] \subseteq \mathbb{C}$. By Step 1, every module $N \in \sigma[M]$ is locally Noetherian and CS, hence by Lemma 2 every local direct summand of N is also a direct summand. Thus by Lemma 6, $\sigma[M]$ is a pure semisimple category. Every module in $\sigma[M]$ is a direct sum of indecomposable (whence uniform) Noetherian modules with local endomorphism rings (see [30]).

Next we show that every uniform module $L \in \sigma[M]$ is quasi-injective. By the above argument, End L is local. Let N be a countable direct sum of copies of L, i.e. $N = \bigoplus_{i\geq 1} L_i$, with $L_i \cong L$ for all i. Since N is locally Noetherian and CS, every local direct summand in N is a direct summand, by Lemma 2. Also, because End L_i is local for each i, the family $\{L_i: i \geq 1\}$ is locally semi-T-nilpotent (see [17, Theorem 7.3.15] or [22, Theorem 2.25]). Let $\theta: L \to L$ be any monomorphism. Suppose that θ is not an isomorphism. By the local semi-T-nilpotency of $\{L_i: i \geq 1\}$, it follows easily that, for any $x \in L$, there is a positive integer n such that $\theta^n(x) = 0$, which implies that $x = 0$, a contradiction. Thus any monomorphism $\theta: L \to L$ is an isomorphism. Since $L \oplus L$ is CS, by Lemma 3(b) it follows that L is quasi-injective.

Now we show that the uniform module L in $\sigma[M]$ is uniserial. Let A and B be any submodules of L. Since A and B have local endomorphism rings and the external direct sum $A \oplus B$ is CS, either B is A-injective or B is embedded in A, by Lemma 3(c). If B is A-injective, then since $C = A \cap B$ is an essential submodule of A, the identity map in C can be extended to a monomorphism from A to B. Thus either A is embedded in B or B is embedded in A.

We may assume that there is a monomorphism $\varphi: A \to B$. Since L is quasi-injective, φ can be extended to a homomorphism $\psi: L \to L$, and clearly ψ is an isomorphism (see above). But A is a quasi-injective essential submodule of L, so it is well-known that $g(A) \subseteq A$ for any homomorphism $g: L \to L$. In particular, this implies that $A = \psi(A) = \varphi(A) \subseteq B$. Hence L is uniserial.

We claim now that length $L \leq 2$. Since L is uniform, we know that L is Noetherian. Suppose that L is not simple, then L contains a nonzero maximal submodule L_1. If L_1 is not simple, then L_1 contains a nonzero maximal submodule L_2. Let L_3 be a (possibly zero) maximal submodule of L_2. Then L/L_3 is a uniserial module of length 3, and, by Lemma 4, the direct sum $(L/L_3) \oplus (L_1/L_2)$ is not a CS-module, a contradiction. This shows that L_1 is simple, and so length $L \leq 2$.

Thus we have shown that every module M in $\mathbb{C}$ is a direct sum of modules of length at most 2. To complete the proof of (a)$\Rightarrow$(b), it remains to show that if $T \in \mathbb{C}$ and length $T = 2$, then T is X-injective for any $X \in \mathbb{C}$. Indeed, X has a decomposition $X = \bigoplus_{\alpha \in \Omega} X_\alpha$, with length $X_\alpha \leq 2$, for each $\alpha \in \Omega$. If X_α is simple, clearly T is X_α-injective. If X_α has length 2, we consider the CS-module $T \oplus X_\alpha$. Let F and G be any direct summands of $T \oplus X_\alpha$ with $F \cap G = 0$, then by the Krull–Schmidt theorem, F and G have length 2, hence $F \oplus G = T \oplus X_\alpha$. It follows that $T \oplus X_\alpha$ is a quasi-continuous module, hence T is X_α-injective. Therefore T is X-injective, as required.

(b) $\Rightarrow$ (c). By (b), for every $M \in \mathbb{C}$, $M = (\bigoplus_{i \in I} N_i) \oplus S$, where each N_i $(i \in I)$ has length 2 and N_i is X-injective for all $X \in \mathbb{C}$, and S is semisimple. Let $N = \bigoplus_{i \in I} N_i$. Then for every module $X \in \mathbb{C}$, since X is locally Noetherian, N is X-injective.

(c) $\Rightarrow$ (a). Let $M \in \mathbb{C}$, M cyclic. Every quotient of a cyclic submodule of M is a direct sum of a quasi-injective module and a semisimple module of finite length. Thus, by [10, Theorem 1.3], M has finite uniform dimension. In particular, $M = M_1 \oplus \cdots \oplus M_n$, where each M_i is cyclic indecomposable, hence each M_i is simple or uniform quasi-injective. If, for $1 \leq i \leq n$, M_i is not simple, then any cyclic proper submodule C_i in M_i is simple because C_i cannot be M_i-injective. Thus each M_i $(1 \leq i \leq n)$ is either simple or of length 2, and M is Noetherian. It follows that every module in $\mathbb{C}$ is locally Noetherian.

Now let K be any module in $\mathbb{C}$. By Zorn's lemma, there exists a maximal family of independent modules $L_\alpha (\alpha \in \Omega)$ in K such that L_α is X-injective for all $X \in \mathbb{C}$. Then $L = \bigoplus_{\alpha \in \Omega} L_\alpha$ is X-injective for all $X \in \mathbb{C}$ (see [22, Theorem 1.11]). Thus $K = L \oplus T$ for some submodule T of K. Take any cyclic submodule D of T; then $D = N \oplus S$, where N is X-injective for all $X \in \mathbb{C}$ and S is semisimple. By the maximality of the family $\{L_\alpha: \alpha \in \Omega\}$, we observe that $N = 0$, whence T is semisimple. Since L is locally Noetherian and quasi-injective, by Lemma 2, $L = \bigoplus_{i \in I} H_i$ with each H_i $(i \in I)$ uniform. By a similar argument as above, if H_i is not simple, then every cyclic proper submodule of H_i is simple, so H_i has length 2. In this case clearly H_i is cyclic and H_i is X-injective for all $X \in \mathbb{C}$. Thus K is a CS-module by Lemma 5. □

Goodearl [14] characterized right SI-rings. In [28, p. 351] the question was raised of characterizing rings all of whose singular right modules are CS. Rizvi and Yousif [29] recently studied a special case, namely rings with all singular right modules quasi-continuous, and they showed that these were precisely the rings with all singular right modules semisimple. Moreover, if R is a right nonsingular ring then every singular right R-module is quasi-continuous if and only if R is a right SI-ring (see [29, Proposition 3.4]).

Since the class of all singular right modules over a ring R is closed under direct sums, quotients and submodules, Theorem 7 gives immediately the following result.

Corollary 8. *Let R be any ring. Then the following statements are equivalent.*

(a) *Every singular right R-module is CS.*

(b) *Every singular right R-module M has a decomposition $M = \bigoplus_{i \in I} M_i$, where each M_i is simple, or M_i has length 2 and is X-injective for each singular right R-module X.*

(c) *Every (cyclic) singular right R-module M has a decomposition $M = N \oplus S$, where N is X-injective for all singular right modules X, and S is semisimple.*

If R is a right nonsingular ring, then the class of all singular right R-modules is closed under essential extensions (see, for example, [15, Proposition 1.23]). In this case, if N is a singular right R-module such that N is X-injective for every singular R-module X, then N is $E(N)$-injective, and it follows that $N = E(N)$, i.e. N is an injective module. This gives at once the following result.

Corollary 9. *Let R be a right nonsingular ring. Then the following statements are equivalent.*

(a) *Every singular right R-module is CS.*

(b) *Every (cyclic) singular right R-module is a direct sum of an injective module and a semisimple module.*

Example 10. Let K be any field and let n be a positive integer. Let $T_n(K)$ denote the ring of all upper triangular $n \times n$ matrices with entries in K. Then $T_n(K)$ is a (right and left) hereditary Artinian serial ring for every positive integer n. If $n = 2$ then $T_n(K)$ is an SI-ring. If $n = 3$ then $T_n(K)$ is not a right SI-ring but every singular right module is CS.

Proof. It is well known that $T_n(K)$ is hereditary Artinian serial, and hence, in particular, nonsingular. If $n = 2$ then $T_n(K)$ is an SI-ring by [14, Theorem 3.11]. Now suppose that $n = 3$. Let $R = T_3(K)$ and let S denote the right socle of R. Then S consists of all matrices in R with first two columns zero, and S is obviously an essential right ideal of R. Note that $R/S \cong T_2(K)$. Thus R is not a right SI-ring, by [14, Theorem 3.11]. However, R/S is an Artinian serial ring with $J(R/S)^2 = 0$. By [9, Theorem 2.6], every right (R/S)-module is a direct sum of an injective module and a semisimple module and thus every (R/S)-module is a CS-module by Theorem 7. It follows that every singular R-module is CS. □

4. Rings whose finitely generated modules are CS

Now we will study rings over which every finitely generated (right) module is CS. By [28, Corollary 1], over such rings every finitely generated module is a direct sum of uniform modules. As was remarked in [28, p.345], the ring $\mathbb{Z}$ of integers is an example to show that the converse is false.

For the next result, we recall that a ring R is called *semiregular* if R/J is a von Neumann regular ring and idempotents can be lifted over the Jacobson radical J. Vanaja and Purav [32, Proposition 2.13] have proved (c)⇔(d) using different methods.

Theorem 11. *The following conditions are equivalent for a ring R with Jacobson radical J.*

(a) *$E(R_R)$ is projective and every 2-generated right R-module is CS.*

(b) *R is semiregular and every 2-generated right R-module is CS.*
(c) *Every right R-module is CS.*
(d) *R is (left and right) Artinian serial and* $J^2 = 0$.
(e) *The left-handed versions of* (a)–(c).

Proof. (a) $\Rightarrow$ (b). Since every cyclic right R-module is CS, by Lemma 1, we have a complete family of orthogonal idempotents $e_1, \ldots, e_n$ of R such that each e_iR is uniform as a right R-module. Consider the injective hull $E(e_iR)$ of e_iR; then $E(e_iR)$ is indecomposable injective and projective, hence there is an idempotent f_i of R such that $E(e_iR) \cong f_iR$ (see, for example, [13, Theorem 20.15]). Because End (f_iR) is local, f_iR contains a unique maximal submodule (namely f_iJ). Thus every quotient of f_iR is indecomposable and CS, so is uniform. Clearly f_iR is a uniserial module. Since e_iR is isomorphic to a submodule of f_iR, e_iR is also uniserial. Therefore R is a right serial ring, so it is well known that R is semiperfect (see, for example, [33, 55.3]). In particular, R is semiregular.

(b) $\Rightarrow$ (d). By Lemma 1, R/J has finite uniform dimension, so R/J is semisimple and hence R is semiperfect. Thus there is a complete set of orthogonal idempotents $e_1, \ldots, e_n$ of R such that each e_iR has local endomorphism ring. Since each quotient of e_iR is CS, we can apply the same argument as in the proof of (a) $\Rightarrow$ (b) to show that each e_iR is uniserial. Thus R is a right serial ring.

Now we claim that each e_iR has nonzero socle. Suppose that $\mathrm{Soc}(e_iR) = 0$ for some $i \geq 1$. Take any simple right R-module U; then the 2-generated module $e_iR \oplus U$ is CS. Since U cannot be embedded in e_iR, U is e_iR-injective by Lemma 3. Thus e_iR is a V-module, so in particular $J(e_iR) = 0$ (see, for example, [33, 23.1]). Hence e_iR is simple, a contradiction. Therefore $\mathrm{Soc}(e_iR) \neq 0$ for each e_iR, so R has finitely generated essential right socle. For any two-sided ideal K of R, R/K is also a right serial ring, and it is easy to check that every 2-generated right (R/K)-module is CS, so by the above argument, R/K has finitely generated essential right socle. Then by a result of Beachy [3] it follows that R is right Artinian.

Suppose that $J^2 \neq 0$. Then there is a positive integer j such that $e_jJ^2 \neq 0$. Since e_jR is uniserial, we have a composition series

$$e_jR \supset e_jJ \supset e_jJ^2 \supset e_jJ^3 \supset \cdots .$$

Then e_jR/e_jJ^3 is uniserial of length 3 and so, by Lemma 4, $(e_jR/e_jJ^3) \oplus (e_jJ/e_jJ^2)$ is not CS, a contradiction. Thus $J^2 = 0$, hence all $e_1R, \ldots, e_nR$ have length ≤ 2. If length $e_kR =$ length $e_tR = 2$, then since $e_kR \oplus e_tR$ is CS, it is easy to see that $e_kR \oplus e_tR$ is quasi-continuous (see the last part of the proof of Theorem 7), and it follows that e_kR and e_tR are relatively injective. Thus, it is clear that if e_kR has length 2, then e_kR is an injective R-module. Now by [9, Theorem 2.6] it follows that R is (left and right) Artinian serial (with $J^2 = 0$).

(d) $\Rightarrow$ (c). If R is (left and right) Artinian serial with $J^2 = 0$, then every R-module is a direct sum of an injective module and a semisimple module (see [9, Theorem 2.6]). Thus, by Theorem 7, every R-module is CS.

(c) $\Rightarrow$ (a). By [24, Theorem II].

(e) $\Leftrightarrow$ (d). By left–right symmetry. □

The ring $T_3(K)$ of upper triangular 3×3 matrices over a field K has the property that every singular module is CS (see above). However, Theorem 11 shows that not every $T_3(K)$-module is CS. For any module M, the *second singular submodule* $Z_2(M)$ is defined as follows:

$$Z_2(M)/Z(M) = Z(M/Z(M)).$$

Lemma 12. *Let R be a ring and M a right R-module. Then M is CS if and only if $M = Z_2(M) \oplus N$, where $Z_2(M)$ and N are CS and $Z_2(M)$ is N-injective.*

Proof. See [19, Theorem 1]. □

Theorem 13. *The following statements are equivalent for a ring R.*

(a) *R is right nonsingular and every 2-generated right R-module is CS.*

(b) *$R = R_1 \oplus \cdots \oplus R_n$ is a direct sum of rings R_i $(1 \le i \le n)$, each Morita equivalent to an upper triangular 2×2 matrix ring over a division ring, or to a simple Noetherian SI-domain.*

Moreover, in this case, every finitely generated right (and left) R-module is CS.

Proof. (a) $\Rightarrow$ (b). Let L be any cyclic singular right R-module, and consider $M = L \oplus R_R$. Then M is a CS-module by hypothesis. Since R is right nonsingular, we have $Z_2(M) = Z(M) = L$, so by Lemma 12, L is R-injective. Thus every cyclic singular right R-module is injective, so by [28, Corollary 5], R is a right SI-ring.

By [14, Theorem 3.11], there is a ring direct decomposition $R = R_0 \oplus B_1 \oplus \cdots \oplus B_m$, such that R_0/S_0 is semisimple, where S_0 is the right socle of the ring R_0, and each B_j $(1 \le j \le m)$ is a simple right Noetherian right SI-ring. Since every cyclic right R-module is CS, R has finite right uniform dimension by Lemma 1, hence R_0 is right Artinian. Since every 2-generated right R_0-module is CS, by Theorem 11 ((b) $\Leftrightarrow$ (d)), R_0 is left and right Artinian serial and $J(R_0)^2 = 0$. Note that R_0 is right nonsingular, and all nonsingular right R_0-modules are projective, hence $R_0 = A_1 \oplus \cdots \oplus A_n$, where each A_i is Morita equivalent to a full upper triangular matrix ring T_i over a division ring D_i (see [15, Theorem 5.28]). Each indecomposable module over R_0 (and hence A_i) has length ≤ 2, thus by [15, Proposition 5.25, Theorem 5.27], it follows easily that each T_i is an upper triangular 2×2 matrix ring over D_i.

Now we consider the simple right Noetherian rings B_j $(1 \le j \le m)$. Let $1 \le j \le m$. Since B_j is right nonsingular and $B_j \oplus B_j$ is CS, it is easy to see that every 2-generated nonsingular right B_j-module is projective. Hence, by the proof of Theorem 5.3 in [21], the right classical quotient ring Q_j of B_j is also the left classical quotient ring of B_j, thus B_j is also a left Goldie ring [6, Theorem 1.28]. Also, since B_j is right hereditary right Noetherian, B_j is left semi-hereditary [6, Corollary 8.19]. Since B_j is right SI, B_j/K_j is semisimple for every essential right ideal K_j of B_j [14, Proposition 3.1], thus B_j is left

Noetherian by [12, Theorem 11]. Since B_j is Morita equivalent to a simple right SI-domain F_j ([14, Theorem 3.11]), it follows that F_j is left Noetherian and hence F_j is left SI by [8, Theorem 6.26].

(b) $\Rightarrow$ (a). Suppose that $R = R_1 \oplus \cdots \oplus R_n$ where the R_i's are as in (b). Since the property of being a right SI-ring is a Morita invariant, each R_i $(1 \le i \le n)$ is (two-sided) SI. Hence R is (two-sided) SI. In particular, R is left and right hereditary [14, Proposition 3.3]. Since every (two-sided) hereditary Noetherian prime ring is CS (see [5, Proposition 6.8]), it follows that each R_i $(1 \le i \le n)$ is left and right CS. Hence R is (two-sided) CS.

Now let M be any finitely generated right R-module. Then $Z(M)$ is injective, so $M = Z(M) \oplus N$, and clearly N is nonsingular finitely generated. Since R is hereditary CS, by [7, Theorem 4.1], every nonsingular finitely generated right R-module is projective, hence it follows easily that every nonsingular finitely generated right R-module is CS. Therefore, N is a CS-module. Thus, by Lemma 12, M is CS. By symmetry, we can show that every finitely generated nonsingular left R-module is CS. This proves Theorem 13. □

Corollary 14. *Let R be a commutative ring. Then the following statements are equivalent.*

(a) *Every 2-generated R-module is CS.*

(b) *Every R-module is CS.*

(c) *$R = R_1 \oplus \cdots \oplus R_n$ is a direct sum of rings R_i $(1 \le i \le n)$, each a QF ring of length 2 or a field.*

Proof. (a) $\Rightarrow$ (b). Let R have prime radical N and Jacobson radical J. Then $\bar{R} = R/N$ is a semiprime commutative ring and so is nonsingular. Clearly every 2-generated $\bar{R}$-module is CS. By the proof of (a) $\Rightarrow$ (b) in Theorem 13, $\bar{R}$ is an SI-ring. Because $\bar{R}$ is commutative, it follows by [14, Theorem 3.9] that $\bar{R}$ is von Neumann regular. But every cyclic R-module is CS, so by Lemma 1, $\bar{R}$ has finite uniform dimension, hence $\bar{R}$ is semisimple. It follows that $N = J$ and R is semiperfect. By Theorem 11, every R-module is CS.

(b) $\Rightarrow$ (c) $\Rightarrow$ (a). Immediate by Theorem 11. □

By [28, Corollary 1], if R is a ring such that every cyclic right R-module is CS, then every cyclic right R-module is a direct sum of uniform modules. As was remarked in [28, p. 345], it would be interesting to know when the converse is true. We observe that if a ring R is a direct sum of uniform right ideals, then for every two-sided ideal I of R which is a complement right ideal of R, $I = eR$ for some idempotent e of R (see [6, Corollary 6.6]). We shall call a ring R a *right duo* ring if every right ideal of R is a two-sided ideal. We have at once:

Proposition 15. *Let R be a right duo ring. Then every cyclic right R-module is CS if and only if every cyclic right R-module is a direct sum of uniform modules.*

Note that if, for a ring R, all cyclic right modules are continuous, then R is a finite direct sum of simple Artinian rings and right duo rings (see [28, Corollary 9]). Next we look briefly at commutative rings whose finitely generated singular modules are CS.

Proposition 16. *Let R be a commutative ring. Then*

(a) *Every cyclic singular R-module is CS if and only if every cyclic singular R-module is a direct sum of uniform modules.*

(b) *Every finitely generated singular R-module is CS if and only if every singular R-module is CS.*

Proof. (a) This follows by the observations before Proposition 15.

(b) Suppose that every finitely generated singular R-module is CS. Let K be any essential ideal of R, and let $R' = R/K$. Let M' be a finitely generated R'-module. Then M' is a finitely generated singular R-module in a natural way. Thus M' is a CS-module over R, so M' is a CS-module over R'. Since R' is commutative, by Corollary 14 and Theorem 11, R' is Artinian serial with $J(R')^2 = 0$.

Now let H be any finitely generated singular R-module. There exists an essential ideal K of R such that $HK = 0$. Thus H has a decomposition $H = H_1 \oplus \cdots \oplus H_m$, where length $H_j \leq 2$ for each $1 \leq j \leq m$.

Let L be any singular R-module of length 2. We claim that L is X-injective for all singular modules X. We suppose first that X is finitely generated and singular. Then $X = \bigoplus_{1 \leq i \leq k} X_i$, where each X_i has length ≤ 2. If X_i is simple, clearly L is X_i-injective. If X_i has length 2, then because $L \oplus X_i$ is finitely generated singular, $L \oplus X_i$ is CS. Then it is easy to prove that $L \oplus X_i$ is quasi-continuous, so L is X_i-injective. Thus L is X-injective for any finitely generated singular R-module X. Now for any singular R-module Y, L is yR-injective for all $y \in Y$, and hence L is Y-injective.

We have shown that every finitely generated singular R-module H has a decomposition $H = E \oplus S$, where E is X-injective for all singular modules X and S is semisimple. By Theorem 7, it follows that every singular R-module is CS. □

Example 17. Let $R = \mathbb{Z}$, the ring of integers. Then every cyclic singular $\mathbb{Z}$-module is quasi-injective (hence CS). However, clearly not every finitely generated singular $\mathbb{Z}$-module is CS (for example $(\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$, p any prime number).

We make one final comment. Let R be any ring. An R-module M is called (*Goldie*) *torsion* if $Z_2(M) = M$. In [29, Theorem 3.10], Rizvi and Yousif prove that the ring R is right SI if and only if every torsion R-module is quasi-continuous. If R is right nonsingular then every torsion module is singular. Thus, for any field K, the ring $T_3(K)$ considered above has the property that every torsion module is CS but $T_3(K)$ is not an SI-ring. Moreover, for any prime p, let S denote the ring $\mathbb{Z}/\mathbb{Z}p^2$. Then the ring T of all upper triangular 2×2 matrices with entries in S has the property that every singular module is CS, but the module T_T is torsion but not CS.

Acknowledgements

This paper was written while the first author was visiting the University of Glasgow and the University of Murcia. He wishes to thank both Departments of Mathematics for kind hospitality, and the Royal Society of London and the Spanish Ministry of Education and Science for their financial support. He would also like to thank Prof. J.L. García for several helpful conversations.

References

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules (Springer, New York, 1973).
[2] Y. Baba and M. Harada, On almost M-projectives and almost M-injectives, Tsukuba J. Math. 14 (1990) 53–69.
[3] J.A. Beachy, On quasi-Artinian rings, J. London Math. Soc. (2) 3 (1971) 449–452.
[4] V. Camillo and M.F. Yousif, CS-modules with *acc* or *dcc* on essential submodules, Comm. Algebra 19 (1991) 655–662.
[5] A.W. Chatters and C.R. Hajarnavis, Rings in which every complement right ideal is a direct summand, Quart. J. Math. Oxford (2) 28 (1977) 61–80.
[6] A.W. Chatters and C.R. Hajarnavis, Rings with Chain Conditions (Pitman, London, 1980).
[7] A.W. Chatters and S.M. Khuri, Endomorphism rings of modules over nonsingular CS-rings, J. London Math. Soc. (2) 21 (1980) 434–444.
[8] J.H. Cozzens and C.C. Faith, Simple Noetherian Rings (Cambridge University Press, London, 1975).
[9] Dinh van Huynh, Nguyen V. Dung and P.F. Smith, Rings characterized by their right ideals or cyclic modules, Proc. Edinburgh Math. Soc. 32 (1989) 355–362.
[10] Dinh van Huynh, Nguyen V. Dung and R. Wisbauer, On modules with finite uniform and Krull dimension, Arch. Math. 57 (1991) 122–132.
[11] Dinh van Huynh, P.F. Smith and R. Wisbauer, A note on GV-modules with Krull dimension, Glasgow Math. J. 32 (1990) 389–390.
[12] C. Faith, On a theorem of Chatters, Comm. Algebra 3 (1975) 169–184.
[13] C. Faith, Algebra II – Ring Theory (Springer, New York, 1976).
[14] K.R. Goodearl, Singular torsion and the splitting properties, Amer. Math. Soc. Memoirs 124 (1972).
[15] K.R. Goodearl, Ring Theory (Marcel Dekker, New York, 1976).
[16] R. Gordon and L.W. Small, Piecewise domains, J. Algebra 23 (1972) 553–564.
[17] M. Harada, Factor categories with applications to direct decomposition of modules, Lecture Notes in Pure and Applied Mathematics, Vol. 88 (Marcel Dekker, New York, 1983).
[18] M. Harada and Y. Sai, On categories of indecomposable modules I, Osaka J. Math. 7 (1970) 323–344.
[19] M.A. Kamal and B.J. Müller, Extending modules over commutative domains, Osaka J. Math. 25 (1988) 531–538.
[20] M.A. Kamal and B.J. Müller, The structure of extending modules over Noetherian rings, Osaka J. Math. 25 (1988) 539–551.
[21] L.S. Levy, Torsion-free and divisible modules over non-integral domains, Canad. J. Math. 15 (1963) 132–151.
[22] S.H. Mohamed and B.J. Müller, Continuous and discrete modules, London Math. Soc. Lecture Note Series, Vol. 147 (Cambrige Univ. Press, Cambrige, 1990).
[23] M. Okado, On the decomposition of extending modules, Math. Japonica 29 (1984) 939–944.
[24] K. Oshiro, Lifting modules, extending modules and their applications to QF-rings, Hokkaido Math. J. 13 (1984) 310–338.
[25] K. Oshiro, On Harada rings I, Math. J. Okayama Univ. 31 (1989) 161–178.
[26] B.L. Osofsky, Rings all of whose finitely generated modules are injective, Pacific J. Math. 14 (1964) 645–650.
[27] B.L. Osofsky, Injective modules over twisted polynomial rings, Nagoya Math. J. 119 (1990) 107–114.

[28] B.L. Osofsky and P.F. Smith, Cyclic modules whose quotients have all complement submodules direct summands, J. Algebra 139 (1991) 342–354.
[29] S.T. Rizvi and M.F. Yousif, On continuous and singular modules, in: Non-commutative Rings Theory, Proceedings, Athens, OH, Lecture Notes in Mathematics, Vol. 1448 (Springer, Berlin, 1989) 116–124.
[30] D. Simson, On pure semisimple Grothendieck categories I, Fund. Math. 100 (1978) 211–222.
[31] B. Stenström, Rings of Quotients (Springer, Berlin, 1975).
[32] N. Vanaja and V.M. Purav, Characterisations of generalised uniserial rings in terms of factor rings, Comm. Algebra 20 (1992) 2253–2270.
[33] R. Wisbauer, Foundations of Module and Ring Theory (Gordon and Breach, Reading, MA, 1991).